New Exact Travelling Wave Solutions of Nonlinear Coagulation Problem with Mass Loss

El-Said A. El-Wakil[a], Essam M. Abulwafa[a], and Mohammed A. Abdou[a,b]

[a] Theoretical Research Group, Physics Department, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt
[b] Faculty of Education for Girls, Physics Department, King Kahlid University, Bisha, Saudia Arabia

Reprint requests to M. A. A.; E-mail: m_abdou_eg@yahoo.com

Z. Naturforsch. **65a,** 209 – 214 (2010); received November 3, 2008 / revised Oktober 7, 2009

This paper suggests a generalized F-expansion method for constructing new exact travelling wave solutions of a nonlinear coagulation problem with mass loss. This method can be used as an alternative to obtain analytical and approximate solutions of different types of kernel which are applied in physics. The nonlinear kinetic equation, which is an integro differential equation, is transformed into a differential equation using Laplace's transformation. The inverse Laplace transformation of the solution gives the size distribution function of the system.

As a result, many exact travelling wave solutions are obtained which include new periodic wave solutions, trigonometric function solutions, and rational solutions. The method is straightforward and concise,and it can also be applied to other nonlinear evolution equations arising in mathematical physics.

Key words: Nonlinear Coagulation Problem; Mass Loss; New Exact Travelling Solutions; Laplace Transform.

1. Introduction

In recent years, nonlinear evolution equations in mathematical physics play a major role in various fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical kinematics, chemical physics, and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction, and convection are very important in nonlinear wave equations.

The investigation of exact solutions of nonlinear evolution equations (NLEEs) plays an important role in the study of nonlinear physical phenomena and gradually becomes one of the most important and significant tasks. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been presented [1 – 15].

Very recently, He and Abdou [2] proposed a straightforward and concise method, called exp-function method, to obtain generalized solitary solutions and periodic solutions of NLEEs. The solution procedure of this method, by the help of Matlab or Mathematica, is of utter simplicity and this method can be easily extended to other nonlinear evolution equations [11, 12].

Coagulation is a very important process in a wide variety of physical, chemical, and biological processes. Consequently, an understanding of its kinetics is of great interest in many problems ranging from colloidal polymer technology to antigenuantibody aggregation and cluster formation in galaxies [16 – 20]. Smoluchowski's equation for rapid coagulation describes the temporal evolution of a system of particles, which are continuously growing as result of pairs of particles coming into contact and bonding to form clusters. Examples of this process include the coagulation of aerosols, colloidal suspension, and the formation of polymers.

The rest of this paper is arranged as follows. Section 2 contains the description of the problem of coagulation. In Section 3, we simply provide the mathematical framework of the extended F-expansion method and also give the solutions of the coagulation problem which include new soliton like solutions and trigonometric function solutions. We conclude the paper in the last section.

2. Description of the Problem

The kinetic equation, which describes the process of coangulation, is called Smoluchowski's equation and is written as [8]

$$\frac{\partial C(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(y,x-y)C(y,t)C(x-y,t)\mathrm{d}y - C(x,t)\int_0^x K(x,y)C(y,t)\mathrm{d}y, \tag{1}$$

where $C(x,t)$ represents the concentration of particles of size x at time t. Size means mass or volume. $K(x,y)$ is the coagulation kernel, which describes the rate at which the particles of sizes x and y coagulate to form a particle of size $x+y$. This kernel is assumed to be symmetric with respect to its arguments. It is non-negative due to its physical interpretation as a probability. The first term on the right hand side of (1) gives the rate of change of particles of size x due to coagulation of particles of size y and xy. The second term represents the depletion of particles of size x by particles coagulating with particles of other size. This problem is interesting for physical systems, which contain conservative mass during coagulation. Systems in which oxidation, melting or evaporation occurs on the exposed surface of the particles during coagulation are interesting examples. Here, the exposed surface of the particle recedes continuously eventually leading to a total loss of the mass of the particle. Thus, (1) with mass loss could be rewritten as [8]

$$\frac{\partial C(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(y,x-y)C(y,t)C(x-y,t)\mathrm{d}y - C(x,t)\int_0^\infty K(x,y)C(y,t)\mathrm{d}y + \frac{\partial}{\partial x}[m(x)C(x,t)], \tag{2}$$

where $m(x)$ is a continuous mass loss rate. The third term on the right hand side of this equation arises when mass is removed continuously from particles of the system.

In the case of constant coagulation as in Smoluchowiski's original coagulation equation, the kernel is taken equal to unity, i. e. $K(x,y) = 1$, so (2) becomes

$$\frac{\partial C(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(y,x-y)C(y,t)C(x-y,t)\mathrm{d}y - C(x,t)\int_0^\infty C(y,t)\mathrm{d}y + m\frac{\partial}{\partial x}[xC(x,t)]. \tag{3}$$

Taking Laplace's transform of (3), we have

$$\frac{\partial u(p,t)}{\partial t} = \frac{1}{2}u^2(p,t) - u(0,t)u(p,t) - mp\frac{\partial u(p,t)}{\partial p}, \tag{4}$$

$$u(p,t) = \int_0^\infty \mathrm{e}^{-py}C(y,t)\mathrm{d}y. \tag{5}$$

The function $u(0,t)$ is given by [8]

$$u(0,t) = \frac{1}{\left[1+\frac{t}{2}\right]}. \tag{6}$$

Then (4) admits to

$$\frac{\partial u(p,t)}{\partial t} = \frac{1}{2}u^2(p,t) - \frac{1}{1+\frac{t}{2}}u(p,t) - mp\frac{\partial u(p,t)}{\partial p}. \tag{7}$$

In the next section, the reduced equation (7) is easier to solve than the original one by means of the extended F-expansion method and many exact travelling wave solutions are obtained which include new periodic wave solutions, trigonometric function solutions, exponentional solutions, and rational solutions.

3. Method and its Applications

Let us consider the nonlinear evolution equations, say in two independent variables x and t, as

$$N(\phi, \phi_x, \phi_t, \phi_{xx}, \ldots) = 0, \tag{8}$$

where N is in general a polynomial in ϕ and its various partial derivatives. Seeking for the travelling wave solution of (8), we take

$$\phi(x,t) = \phi(\xi), \quad \xi = k(x+\lambda t), \tag{9}$$

where k and λ are constants to be determined later. Inserting (9) into (8) yields an ordinary differential equation (ODE) for $\phi(\xi)$

$$\psi(\phi, k\phi', \lambda k\phi', k^2\phi'', \ldots) = 0. \tag{10}$$

The next crucial step is to express the solution we are looking for in the general form

$$\phi(\xi) = a_0 + \sum_{i=1}^{N}[a_iF^i(\xi) + b_iF^{-i}(\xi) + c_iF^{i-1}(\xi)F'(\xi) + d_iF^{-i}(\xi)F'(\xi)], \tag{11}$$

where $a_0 = a_0(x)$, $a_i = a_i(x)$, $b_i = b_i(x)$, $c_i = c_i(x)$, $d_i = d_i(x) (i = 1,2,\ldots,n)$. N is a positive integer that can be determined by balancing the hightest-order linear term with the nonlinear terms in the equation. $F(\xi)$ and $F^{'}(\xi)$ satisfy the Riccati equation

$$F^{'}(\xi) = A + BF(\xi) + CF^2(\xi), \tag{12}$$

where A, B, and C are constants to be determined. Inserting (11) into (10) and with the aid of (12), the left hand side of (10) can be converted into a finite series in $F^{'i}(\xi)F^j(\xi)$. Equating each coefficient of $F^{'i}(\xi)F^j(\xi)(i=0,1;j=0,\pm1,\pm2,\ldots)$ to zero yields a system of algebraic equations for a_0, a_i, b_i, c_i, $d_i(i=1,2,\ldots,n)$. The solutions of this system can be expressed by A, B, C. Substituting these results into (11), we can obtain the general form for travelling wave solutions of (1). With the aid of the Appendix A, from the general form of the travelling wave solutions, we can give a series of soliton-like solutions, trigonometric function solutions, and rational solutions of (1).

4. New Exact Solutions of Nonlinear Coagulation Problem with Mass Loss

To look for the travelling wave solution of (7), we make the transformation

$$\xi = kp + \alpha t, \tag{13}$$

where k and α are constants to be determined later. Then (7) reduces to

$$[\alpha + mpk]u^{'}(\xi) - \frac{1}{2}u^2(\xi) + \left[\frac{1}{1+\frac{t}{2}}\right]u(\xi) = 0. \tag{14}$$

Our gool in this paper is to solve (14) by the extended F-expansion method mentioned above. Considering the homogeneous balance between $u^2(\xi)$ and $u^{'}(\xi)$ in (14), yields $N=1$, so we suppose that the solution of (1) can be expressed by

$$\begin{aligned} u(\xi) = a_0 + \big[&a_1F(\xi) + b_1F^{-1}(\xi) \\ &+ c_1F^{'}(\xi) + d_1F^{-1}(\xi)F^{'}(\xi)\big], \end{aligned} \tag{15}$$

where a_0, a_1, b_1, c_1, and d_1 are constants to be determined later, $F(\xi)$ and $F^{'}(\xi)$ satisfy (12).

Substituting (15) along with (12) into (14), then the left hand side of (14) is converted into a polynomial of $F^{'i}(\xi)F^j(\xi)(i=0,1;j=0,\pm1,\pm2,\ldots)$. Setting each coefficients to zero, we get a set of over-determined algebraic equations for a_0, a_1, b_1, c_1, d_1, and k. Solving this system using Maple, we get the following solution:

Case A: When $A=0$, we have

$$\begin{aligned} &b_1 = 0, \quad c_1 = 0, \quad d_1 = d_1, \\ &a_1 = -\frac{C(-4+2d_1B+d_1tB)}{B(t+2)}, \\ &a_0 = -\frac{(-4+2d_1B+d_1tB)}{t+2}, \\ &k = -\frac{\alpha Bt + 2\alpha B - 2}{mpB(t+2)}. \end{aligned} \tag{16}$$

Case B: When $B=0$, we have

$$\begin{aligned} &b_1 = -d_1A, \;\; c_1 = 0, \;\; d_1 = d_1, \;\; a_0 = \frac{2}{t+2}, \\ &k = -\frac{-2C\alpha At - 4AC\alpha + 2\sqrt{-C}A}{2CA(2+t)pm}, \\ &a_1 = \frac{C(-2d_1 - d_1t + 2kpm(2+t) + 2\alpha t + 4\alpha)}{2+t}. \end{aligned} \tag{17}$$

Case C: When $A=B=0$, we have

$$\begin{aligned} &b_1 = 0, \quad c_1 = 0, \quad d_1 = d_1, \quad a_0 = \frac{4}{t+2}, \\ &k = k, \quad a_1 = -Cd_1. \end{aligned} \tag{18}$$

Inserting these solutions into (15), with the aid of Appendix A, we have many soliton-like solutions, trigonometric function solutions, and rational solutions of (1) as follows:

(I): For $A=0, B=1, C=-1$, from the Appendix A, then $F(\xi) = \frac{1}{2} - \frac{1}{2}\tanh(\xi)$. By case (A), we have soliton-like solutions of (1):

$$\begin{aligned} u_1(\xi) = &-\frac{(-4+2d_1+d_1t)}{t+2} \\ &+ \frac{(-4+2d_1+d_1t)}{(t+2)}\left[\frac{1}{2} - \frac{1}{2}\tanh(\xi)\right] \\ &+ d_1\left[\frac{1}{2} + \frac{1}{2}\tanh(\xi)\right]^{-1}\left[\frac{1}{2} - \frac{1}{2}\tanh^2(\xi)\right], \\ \xi = &\left[-\frac{\alpha t + 2\alpha - 2}{mp(t+2)}\right]p + \alpha t. \end{aligned} \tag{19}$$

(II): In case of $A=0, B=-1, C=1$, from the Appendix A, then $F(\xi) = \frac{1}{2} - \frac{1}{2}\coth(\xi)$. By case (A), we have soliton-like solutions of (1):

$$\begin{aligned} u_2(\xi) = d_1 &- \frac{(-4+2d_1+d_1t)}{(t+2)}\left[\frac{1}{2} - \frac{1}{2}\coth(\xi)\right] \\ &+ d_1\left[\frac{1}{2} - \frac{1}{2}\coth(\xi)\right]^{-1}\left[-\frac{1}{2} + \frac{1}{2}\coth^2(\xi)\right], \end{aligned} \tag{20}$$

$$\xi = \left[-\frac{\alpha t + 2\alpha - 2}{mp(t+2)}\right] p + \alpha t$$

(III): If $A = \frac{1}{2}, B = 0, C = -\frac{1}{2}$ from the Appendix A, then $F(\xi) = \coth(\xi) + \operatorname{csch}(\xi)$ or $F(\xi) = \tanh(\xi) \pm \mathrm{i}\operatorname{sech}(\xi)$. By case (B), we have soliton-like solutions of (1):

$$u_3(\xi) = \frac{2}{2+t} + \frac{(-2+2d_1+d_1t)[\coth(\xi)+\operatorname{csch}(\xi)]}{2(2+t)} - \frac{2+2d_1+d_1t}{2(2+t)[\coth(\xi)+\operatorname{csch}(\xi)]} + \frac{d_1[1-\coth^2(\xi)-\operatorname{csch}(\xi)\coth(\xi)]}{[\coth(\xi)+\operatorname{csch}(\xi)]}, \quad (21)$$

$$u_4(\xi) = \frac{2}{2+t} + \frac{(-2+2d_1+d_1t)[\tanh(\xi)+\mathrm{i}\operatorname{sech}(\xi)]}{2(2+t)} - \frac{2+2d_1+d_1t}{2(2+t)[\tanh(\xi)+i\operatorname{sech}(\xi)]} + \frac{d_1[1-\tanh^2(\xi)-\mathrm{i}\operatorname{sech}(\xi)\tanh(\xi)]}{[\tanh(\xi)+\mathrm{i}\operatorname{sech}(\xi)]}, \quad (22)$$

$$\xi = \left[-\frac{2\alpha+\alpha t-\mathrm{i}}{mp(2+t)}\right] p + \alpha t.$$

(IV): For $A = 1$, $B = 0$, $C = -1$ from the Appendix A, then $F(\xi) = \tanh(\xi)$. By case (B), admits to soliton-like solutions of (1):

$$u_5(\xi) = \frac{2}{t+2} + \frac{2d_1+d_1t+1}{2+t}\tanh(\xi) - \frac{-1+2d_1+d_1t}{(2+t)\tanh(\xi)} + \frac{d_1(1-\tanh^2(\xi))}{\tanh(\xi)}, \quad (23)$$

$$u_6(\xi) = \frac{2}{t+2} + \frac{(2d_1+d_1t+1)}{(2+t)}\coth(\xi) - \frac{(-1+2d_1+d_1t)}{(2+t)\coth(\xi)} + \frac{d_1(1-\coth^2(\xi))}{\coth(\xi)}, \quad (24)$$

$$\xi = \left[-\frac{2\alpha t+1+4\alpha}{mp(2+t)}\right] p + \alpha t.$$

(V): When $A = C = \frac{1}{2}$, $B = 0$, with Appendix A, then $F(\xi) = \sec(\xi) + \tan(\xi)$ or $\csc(\xi) - \cot(\xi)$. By case (B), we have trigonometric function solutions of (1):

$$u_7(\xi) = \frac{2}{2+t} + \frac{(2\mathrm{i}-2d_1-d_1t)(\sec(\xi)+\tan(\xi))}{2(2+t)} - \frac{(2\mathrm{i}+2d_1+d_1t)}{2((2+t)(\sec(\xi)+\tan(\xi))} + \frac{d_1(\sec(\xi)\tan(\xi)+1+\tan^2(\xi))}{(\sec(\xi)+\tan(\xi))}, \quad (25)$$

$$u_8(\xi) = \frac{2}{(2+t)} + \frac{(2\mathrm{i}-2d_1-d_1t)(\csc(\xi)-\cot(\xi))}{2(2+t)} - \frac{(2\mathrm{i}+2d_1+d_1t)}{2((2+t)(\csc(\xi)-\cot(\xi))} + \frac{d_1(-\csc(\xi)\cot(\xi)+1+\cot^2(\xi))}{\csc(\xi)-\cot(\xi)}, \quad (26)$$

$$\xi = \left[\frac{(\mathrm{i}-2\alpha-\alpha t)}{(2+t)mp}\right] p + \alpha t.$$

(VI): In case of $A = C = -\frac{1}{2}$, $B = 0$, with the aid of Appendix A, then $F(\xi) = \sec(\xi) - \tan(\xi)$ or $\csc(\xi) + \cot(\xi)$. By case (B), we have trigonometric function solutions of (1):

$$u_9(\xi) = \frac{2}{2+t} - \frac{(4\mathrm{i}-2d_1-d_1t)[\sec(\xi)-\tan(\xi)]}{2(2+t)} + \frac{d_1}{2[\sec(\xi)-\tan(\xi)]} + \frac{d_1[\sec(\xi)\tan(\xi)-1-\tan^2(\xi)]}{[\sec(\xi)-\tan(\xi)]}, \quad (27)$$

$$u_{10}(\xi) = \frac{2}{2+t} - \frac{(4\mathrm{i}-2d_1-d_1t)[\csc(\xi)+\cot(\xi)]}{2(2+t)} + \frac{d_1}{2[\csc(\xi)+\cot(\xi)]} + \frac{d_1[-\csc(\xi)\cot(\xi)-1-\cot^2(\xi)]}{[\csc(\xi)+\cot(\xi)]}, \quad (28)$$

$$\xi = \left[\frac{2\mathrm{i}-2\alpha-\alpha t}{mp(2+t)}\right] p + \alpha t.$$

(VII): In the limiting case $A = C = 1$, $B = 0$, from the Appendix A, then $F(\xi) = \tan(\xi)$. By case (B), admits to a trigonometric function solution of (1):

$$u_{11}(\xi) = \frac{2}{2+t} + \frac{(-2d_1 - d_1t + \mathrm{i})\tan(\xi)}{2+t} - \frac{\mathrm{i} + 2d_1 + d_1t}{(2+t)\tan(\xi)} + \frac{d_1(1+\tan^2(\xi))}{\tan(\xi)}, \quad (29)$$

$$\xi = \left[\frac{\frac{\mathrm{i}}{2} - \alpha t - 2\alpha}{(2+t)pm}\right] p + \alpha t.$$

(VIII): When $A = C = -1$, $B = 0$, from the Appendix A, then $F(\xi) = \cot(\xi)$. By case (B), we have a trigonometric function solution of (1):

$$u_{12}(\xi) = \frac{2}{2+t} - \frac{(-2d_1 - d_1t + \mathrm{i})\cot(\xi)}{2+t} + \frac{\mathrm{i} + 2d_1 + d_1t}{(2+t)\cot(\xi)} + \frac{d_1(-1-\cot^2(\xi))}{\cot(\xi)}, \quad (30)$$

$$\xi = \left[\frac{\frac{\mathrm{i}}{2} - \alpha t - 2\alpha}{(2+t)pm}\right] p + \alpha t.$$

(IX): For $A = B = 0$, $C \neq 0$, from the Appendix A, then $F(\xi) = -\frac{1}{C\xi+\lambda}$, admits to a rational solution as

$$u_{13}(\xi) = \frac{4}{2+t}, \quad (31)$$

$$\xi = kp + \alpha t.$$

(IIX): For $B = C = 0$, $A \neq 0$, from the Appendix A, then $F(\xi) = A\xi$, admits to a rational solution as follows:

$$u_{14}(\xi) = -\frac{2a_0 + a_0t - 4}{2+t} + a_0, \quad (32)$$

$$\xi = kp + \alpha t.$$

(IIIX): As long as $C = 0$, A, $B \neq 0$, from the Appendix A, then $F(\xi) = \frac{\mathrm{e}^{Bt} - A}{B}$, admits to an exponentional solution as follows:

$$u_{15}(\xi) = a_0 + \frac{(2a_0 + a_0t - 4 + Btd_1 + 2Bd_1)[\mathrm{e}^{Bt} - A]}{A(2+t)} - \frac{A(-4 + 2d_1B + Btd_1)}{(2+t)[\mathrm{e}^{Bt} - A]} - \frac{(2a_0 + a_0t - 4 + Btd_1 + 2Bd_1)\mathrm{e}^{Bt}}{A(2+t)} + \frac{Bd_1\mathrm{e}^{Bt}}{\mathrm{e}^{Bt} - A}, \quad (33)$$

$$\xi = \left[-\frac{Bt\alpha + 2\alpha B + 2}{mpB(2+t)}\right] p + \alpha t.$$

Appendix A

Table. Relations between values of A, B, C, and corresponding $F(\xi)$ in the nonlinear ODE $F'(\xi) = A + BF(\xi) + CF^2(\xi)$.

A	B	C	$F(\xi)$
0	1	-1	$F(\xi) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{\xi}{2})$
0	-1	1	$F(\xi) = \frac{1}{2} - \frac{1}{2}\coth(\frac{\xi}{2})$
$\frac{1}{2}$	0	$-\frac{1}{2}$	$F(\xi) = \coth(\xi) \pm \mathrm{csch}(\xi)$, $\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)$
1	0	-1	$F(\xi) = \tanh(\xi)$, $\coth(\xi)$
$\frac{1}{2}$	0	$\frac{1}{2}$	$F(\xi) = \sec(\xi) + \tan(\xi)$, $\csc(\xi) - \cot(\xi)$
$-\frac{1}{2}$	0	$-\frac{1}{2}$	$F(\xi) = \sec(\xi) - \tan(\xi)$, $\csc(\xi) + \cot(\xi)$
$1(-1)$	0	$1(-1)$	$F(\xi) = \tan(\xi)$, $\cot(\xi)$
0	0	$\neq 0$	$F(\xi) = \frac{-1}{C\xi+\lambda}$
constant	0	0	$F(\xi) = A\xi$
constant $\neq 0$	0	$F(\xi) = \frac{\exp(B\xi)-A}{B}$	

5. Conclusion

In this study, we implement a new analytical technique, namely, a generalized F-expansion method, which is generalized one step further by introducing a new generalized ansatz (11) with a computerized symbolic computation for solving nonlinear equations and an absolutely special form of nonlinear coagulation problem with mass loss arising in physics.

The nonlinear kinetic equation, which is an integro differential equation, is transformed into a differential equation using Laplace's transformation. The inverse Laplace transformation of the solution gives the size distribution function of the system.

As a result, many exact travelling wave solutions are obtained which include new soliton-like solutions, trigonometric function solutions, and rational solutions. It seems that the generalized F-expansion method is more effective and simple than other methods and a lot of solutions can be obtained in the same time. In our work, we use the Maple package.

Finally, it is worthwhile to mention that the method is straightforward and concise, and it can also be applied to other nonlinear evolution equations in physics. This is our task in the future work.

Acknowledgement

The authors would like to express sincerely thanks to the referees for their useful comments and discussions.

[1] J. H. He, Int. J. Modern Phys. B **20**, 1141 (2006).
[2] J. H. He and M. A. Abdou, Chaos, Solitons, and Fractals, **34**, 1421 (2007).
[3] J. H. He, Non perturbative method for strongly nonlinear problems, dissertation.de, Verlag im Internet GmbH, Berlin 2006.
[4] E. M. E. Zayed, and K. A. Gepreel, Appl. Math. Comput. **212**, 1 (2009).
[5] H. Zhang, Commun. Nonlinear Sci. Numer. Simul. **14**, 220 (2009).
[6] M. A. Abdou and A. Elhanbaly. Commun. Nonlinear Sci. Numer. Simul. **12**, 1229 (2007).
[7] S. A. El-Wakil and M. A. Abdou, Z. Naturforsch. **63a**, 1 (2008).
[8] E. M. Abulwafa, M. A. Abdou, and A. A. Mahmoud, Chaos, Solitons, and Fractals **29**, 313 (2006).
[9] M. A. Abdou, Int. J. Mod. Phys. B (in press).
[10] M. A. Abdou, Phys. Lett. A **366**, 61 (2007).
[11] S. A. El-Wakil, M. Madkour, and M. A. Abdou, Phys. Lett. A **369**, 62 (2007).
[12] M. A. Abdou and E. M. Abulwafa, Z. Naturforsch. **63a**, 646 (2008).
[13] M. A. Abdou, Int. J. Nonlinear Sci. **5**, 1 (2008).
[14] M. A. Abdou and S. Zhang, Commun. Nonlinear Sci. Numer. Simul. **14**, 2 (2009).
[15] Y. Peng J. Phys. Soc. Jpn. **72**, 1356 (2003).
[16] S. K. Friedlander, Smoke, dust and haze, New York, Wiley 1977.
[17] J. Silk, Formation Geneva observatory, Switzerland, Sauvering 1980.
[18] D. Johnston and G. Benedek, in: Kinetics of aggregation and gelation. (Ed. F. Family and D. P. Landau), North Holland, Amsterdam 1984.
[19] P. Singh and G. J. Rodgers. J. Phys. A: Math. Gen. **29**, 1437 (1996).
[20] D. O. Pushkin and H. Aref, Phys. Fluids **14**, 694 (2002).